GENERALIZED NUMERICAL SOLUTION OF HEAT EQUATION WITH MOVING BOUNDARY CONDITION

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A numerical method of generalized solution of the unsteady heat transfer involving a moving boundary in a plate, cylinder and sphere is presented. The problem has been solved with the assumption of negligible hydrodynamic velocity but the latent heat, the sensible heats and the different (but constant) thermal properties of both phases, any symmetric initial temperature distribution and variable heat transfer coefficient are taken into account. The combination of the Binder–Schmidt explicite method with the implicite method for the central plane (axis, center) has been used, what warrants the stability of the computation with the use of the same great modulus $M = 0{\cdot}5$ for all three geometries considered. The dependence of the heat transfer area on the radius in the cylinder and sphere during the solidification is taken into account.

The transient cooling of a body and the moving interface problem are of great interest from theoretical as well as practical point of view, and have been amply studied. Many papers have appeared in the scientific literature, in which the authors attempt to solve this problem. The process can be formulated by a set of partial differential equations but their analytical solution is not possible; even the numerical solution is not simple and easy. That is the reason of various simplifying assumptions made by the authors to get the time of freezing and the rate of interface movement. A very good review of the past work relating to moving boundary studies has been reported by Longwell[1] and Tao[2] and is not repeated here. Longwell[1] proposed a very interesting graphical method to include the heat capacity and the different thermal properties of both phases and solved the moving interface problem for any initial temperature distribution in the liquid. His graphic explicite difference solution has not been generalized for different body shapes and is not suitable for numerical computation; his way of using various radial differences in each phase would make the eventual computation difficult. Tao[2] has proposed an excellent generalized numerical solution of freezing but for the saturated liquid only and his way of computation the

temperature in the axis (center) and the freezing time of the central shell in cylindrical (spherical) geometry is not obvious. The present paper attempts to solve the problem of cooling and freezing in a generalized form suitable for numerical computation and without limitation to the saturated liquid, and thus to combine the outstanding features of both Longwell and Tao's work.

THEORETICAL

The cooling and solidifying of an infinite plate of finite thickness, infinitely long cylinder and sphere is described by the general Fourier equation of heat conduction:

$$\frac{\partial \Theta}{\partial \tau} = a\left(\frac{\partial^2 \Theta}{\partial R^2} + \frac{b}{R}\frac{\partial \Theta}{\partial R}\right). \tag{1}$$

The whole process can be devided in three sections. In the first section the liquid body cools, in the second it cools and solidifies and in the last section cools the solid body. (The solution is valid for the heating and melting, too, but for the sake of simplicity only cooling and solidifying will be spoken about.)

For the solution, of the problem, it is assumed that the material has definite, sharp melting point, *i.e.* that it solidifies at a single, unit temperature. Further, it is postulated, that there is no free convection in the liquid phase during its cooling. Under these assumptions the boundary conditions of the problem are defined by the following equations: the temperature of the cooling medium

$$\Theta_{R \gg R_0,\tau} = \Theta_p\,, \tag{2}$$

the symmetry condition

$$(\partial\Theta/\partial R)_{0,\tau} = 0\,, \tag{3}$$

convective heat transfer from the body to the medium

$$-k(\partial\Theta/\partial R)_{R_0,\tau} = h(\Theta_{R_0,\tau} - \Theta_p)\,, \tag{4}$$

the rate of the moving boundary

$$-k_f(\partial\Theta/\partial R)_{\eta,\tau,f} + k_g(\partial\Theta/\partial R)_{\eta,\tau,g} = L\varrho_g(\mathrm{d}\eta/\mathrm{d}\tau)\,, \tag{5}$$

the temperature at the interface of both phases

$$\Theta_{\eta,\tau,f} = \Theta_{\eta,\tau,g} = \Theta_s = \text{const.} \tag{6}$$

Eq. (*1*) and the boundary conditions are made dimensionless by introducing the

dimensionless temperature t, time T and coordinate r:

$$\frac{\partial t}{\partial T} = \frac{\partial^2 t}{\partial r^2} + \frac{b}{r}\frac{\partial t}{\partial r}, \tag{1a}$$

$$t_{r \geq 1, T} = 0, \tag{2a}$$

$$(\partial t/\partial r)_{0,T} = 0, \tag{3a}$$

$$(\partial t/\partial r)_{1,T} = -\mathrm{Bi} \times t_{1,T}, \tag{4a}$$

$$(\partial t/\partial r)_{\mathrm{c},T,\mathrm{g}} = (k_\mathrm{f}/k_\mathrm{g})(\partial t/\partial r)_{\mathrm{c},T,\mathrm{f}} = K(\mathrm{d}e/\mathrm{d}T_\mathrm{g}), \tag{5a}$$

$$t_{\mathrm{c},T,\mathrm{f}} = t_{\mathrm{c},T,\mathrm{g}} = t_\mathrm{s} = \mathrm{const}. \tag{6a}$$

The use of T_g in Eq. (*5a*) emphasizes that the dimensionless time is formed with the thermal diffusivity of the phase g.

The solution of this system of equations is not straightforward and it will be solved by finite difference method. For the solution the explicit Binder–Schmidt[3,4] difference method has been used in combination with the implicit method for the central shells. The body is divided by $n = N$ areas parallel with the surface of the body into N shells of thickness Δr and the time of cooling into k identical intervals ΔT. The dividing areas are counted from the central plane (axis, center), that has the number $n = 0$, to the surface of the body. Their coordinates are then

$$r_n = n\,\Delta r. \tag{7}$$

The difference form of the Eq. (*1a*) is then

$$\frac{\Delta t_T}{\Delta T} = \frac{\Delta^2 t_r}{\Delta r^2} + \frac{b}{r}\frac{\Delta t_r}{\Delta r}, \tag{8}$$

where

$$\Delta t_T = t_{n,k} - t_{n,k-1}, \tag{9}$$

$$\Delta t_r = 0{\cdot}5(t_{n+1} - t_{n-1}), \tag{10}$$

$$\Delta^2 t_r = t_{n+1} + t_{n-1} - 2t_n. \tag{11}$$

By substitution from Eqs (*7*, *9–11*) into the Eq. (*8*) and rearranging, the Eq. (*8*) becomes

$$t_{n,k} = Mt_{n+1,k-1}\left(1 + \frac{b}{2n}\right) + Mt_{n-1,k-1}\left(1 - \frac{b}{2n}\right) + t_{n,k-1}(1 - 2M), \quad (12)$$

where

$$M = \Delta T/\Delta r^2 . \quad (13)$$

The Eq. (*12*) gives the temperature–time dependence in the inside of the body during its cooling. To get the temperature in the central plane (axis, center) and on the surface of the body, the boundary conditions (Eq. (*3*) and Eq. (*4*)) must be taken into account.

For the central plane (axis, center) it follows from Eq. (*1a*)

$$\left(\frac{\partial t}{\partial T}\right)_{0,T} = \lim_{r\to 0}\left(\frac{\partial^2 t}{\partial r^2} + \frac{b}{r}\frac{\partial t}{\partial r}\right) = (1 + b)\left(\frac{\partial^2 t}{\partial r^2}\right)_{0,T} . \quad (14)$$

This equation could be transformed into the explicit difference equation by the same way as the Eq. (*1a*), but to guarantee the stability of the solution even for the highest value of $M = 0{\cdot}5$, it is advantageous to use the implicit form (see *e.g.* Sauljev[5]):

$$\frac{t_{0,k} - t_{0,k-1}}{\Delta T} = (1 + b)\frac{t_{1,k} - 2t_{0,k} + t_{-1,k}}{\Delta r^2} . \quad (15)$$

In the combination with the explicit solution for the other shells it is possible to use this equation in explicit form, too. From the boundary condition (Eq. (*3a*)) it follows that $t_{1,k} = t_{-1,k}$; substitution of the quantity M (Eq. (*13*)) leeds then to the explicit equation

$$t_{0,k} = \frac{2M(1 + b)\,t_{1,k} + t_{0,k-1}}{1 + 2M(1 + b)} . \quad (16)$$

The solution of this Eq. is stable for $M = 0{\cdot}5$, whereas in the case of explicit formulation it would be necessary to use $M \leqq 0{,}25$ to secure the solution stability, what would prolong the computation considerably.

The boundary condition (Eq. (*4a*)) defines the slope of the tangent to the temperature distribution on the surface of the body. This tangent must go through the point that defines the temperature on the surface of the body, *i.e.* the point with the coordinates $r = 1$ and $t_{r=1}$; its equation is therefore

$$t - t_{r=1,T} = -\mathrm{Bi} \times t_{r=1,T}(r - 1) . \quad (17)$$

and it goes through the point $P(r = 1 + 1/\mathrm{Bi};\ t = 0)$. Any tangent to the temperature distribution curve on the surface of the body must go through this point.

The further solution of the boundary condition (Eq. (*4a*)) depends on the way in which the body has been divided. As the most suitable is recommended such a division, that ends in the distance of half an interval Δr outside the surface of the body. This kind of division gives the highest accuracy of the computed surface temperature (and thus of the temperature distribution in the whole body). A great disadvantage of this way of division is the dependence of the number of shells on the Biot number (see Eq. (*21*)). That means, that it is necessary to use a great number of shells in the case of a great Bi, what makes the computation long and expensive. The other kind of division does not use the fictious half-interval and ends on the surface of the body. The solution is not so exact but there is no dependence of the number of shells on the Bi and the cases with a great Bi(Bi > 100) can easily be solved in a reasonable time. The boundary condition (Eq. (*4a*)) has been solved for both ways of division, from which each is suitable for another group of problems.

Dividing the body in the first way, the division ends on the fictious area $n = N$ in the distance 0·5 Δr outside the surface of the body. Then it is

$$\Delta r = 1/(N - 0{\cdot}5) \tag{18}$$

and the coordinate of any dividing area n is

$$r_n = n/(N - 0{\cdot}5)\,. \tag{19}$$

The line (Eq. (*17*)) intersects the fictious area $n = N$ in the point that gives the fictious temperature on this area. This temperature is defined by the substitution of the equation of the area $n = N$, *i.e.* $r_n = N/(N - 0{\cdot}5)$, into the Eq. (*17*):

$$t_{N,k} = t_{N-0{\cdot}5,k}(2N - 1 - \mathrm{Bi})/(2N - 1)\,. \tag{20}$$

By this equation the minimum number $N_{\min}$ of shells, that allows the solution of the problem, is defined:

$$N_{\min} = 0{\cdot}5(\mathrm{Bi} + 1)\,. \tag{21}$$

As the first step of the solution one determines from the Eq. (*20*) the temperature on the fictious area in the time $T = 0$. Using this point and the known initial temperature distribution in the body, it is possible following the Eqs (*12*) and (*16*) to compute the temperatures on the areas $n = 0$ till $n = N - 1$ in the time interval $k = 1$. The tangent to the temperature curve in this and in all further steps is approximated by the line connecting the point t_{N-1} with the point P. The intersection of this line with the surface and the fictious area determines the surface temperature $t_{N-0,5}$ and the temperature t_N on the fictious area:

$$t_{N-0\cdot5,k} = t_{N-1,k}(2N - 1)/(2N - 1 + \mathrm{Bi}) , \tag{22}$$

$$t_{N,k} = t_{N-1,k}(2N - 1 - \mathrm{Bi})/(2N - 1 + \mathrm{Bi}) . \tag{23}$$

The solution in the other case of division, that ends on the surface of the body, is quite a similar one. In this case it is

$$\Delta r = 1/N , \tag{24}$$

$$r_n = n/N . \tag{25}$$

The surface temperature lies on the line connecting the temperature t_{N-1} with the point P:

$$t_{N,k} = t_{N-1,k}N/(N + \mathrm{Bi}) . \tag{26}$$

Having calculated the temperature distribution in the body and on its surface (Eqs (*12*, *16*) and (*22*) or (*26*)), resp. on the fictious area (Eq. (*23*)) in the time interval k, one can continue the calculation in the time interval $k + 1$ and so on until the moment, when the surface temperature reaches the temperature of solidification or some lower temperature. If the last time increment to reach this temperature is fractional, a proportional adjustment of all the last temperature increments is also made.

In such way, the temperature distribution is determined in the moment, when the surface of the body reaches the temperature of solidifying and the body begins to solidify. In this moment the latent heat begins to be released and the solid–liquid interface appears. For the computation of the time dependence of the temperature distribution, the boundary conditions (*5a*) and (*6a*) must be taken into account.

The increment of the interface coordinate de in the Eq. (*5a*) can be defined in the difference form as the ratio of the volume increment ΔV of the new phase g to its surface F and the difference form of the Eq. (*5a*) is

$$(\Delta t/\Delta r)_{e,T,g} - (k_f/k_g)(\Delta t/\Delta r)_{e,T,f} = K\,\Delta V/F\,\Delta T_g . \tag{27}$$

The volume increment ΔV in the time interval ΔT_g is small and unknown. Therefore, the volume increment of phase g, that has the thickness Δr of one shell, will be used for computation. The time of solidification of this shell is computed as a multiple $P\,\Delta T_g$ of the time interval used. On the left side of the Eq. (*27*), the mean temperature difference during all P time intervals is used, what leads to the relation

$$\sum_{j=1}^{P} \left[(\Delta t/\Delta r)_{i,j,g} - (k_f/k_g)(\Delta t/\Delta r)_{i,j,f}\right] = K\,\Delta V/F\,\Delta T_g , \tag{28}$$

where the coordinate e of the interface has been substituted by the index i, which means the number of the inner area of the solidifying shell. In the Eq. (*28*), the temperature differences Δt_i next to the interface are defined by the equations.

$$\Delta t_{i,j,f} = t_{i-1,j} - t_i \,, \quad \Delta t_{i,j,g} = t_i - t_{i+1,j} \,. \tag{29}$$

Including Eqs (*29*) and (*13*) into Eq. (*28*), it becomes

$$\sum_{j=1}^{P} \left[t_i + (k_f/k_g)(t_i - t_{i-1,j}) - t_{i+1,j} \right] = K \, \Delta V/\Delta r F M_g \,. \tag{30}$$

This equation is valid for all shells except the central one; during the last step of solidifying, when the interface reaches the place with coordinate $r = 0$, the liquid phase disappears and the Eq. (*30*) reduces to

$$\sum_{j=1}^{P} (t_i - t_{1,j}) = K \, \Delta V/\Delta r F M_g = CK/M_g \,. \tag{31}$$

The left sides of the Eqs (*30*) and (*31*) are generaly valid for all three types of bodies concerned. The value of the ratio $\Delta V/\Delta r F$ depends on the form of the body and on the position of the solidifying shell, however. For the first shell its value has been designated as C (Eq. (*31*)), for the shells $n = 2$ to $n = N - 1$ as B and for the last shell $n = N$ as A.

To reach high accuracy of the solution, it is advantageous to define the volume increment and its surface area in the difference form that approaches the real conditions.

a) *Plate.* For shells $n = 1$ to $n = N - 1$ is $\Delta V = F \, \Delta r = \text{const.}$ and $F = \text{const.}$, so that $C = B = 1$. If the body is divided without the fictious half-shell, the same is valid for the N-th shell, too: $A = B = 1$. In the other case the thickness of the N-th shell is only a half of the thickness of other shells and $A = 0{\cdot}5$.

b) *Infinite cylinder.* The volume increment of the phase g (for the unit length of the cylinder) is

$$\Delta V/L = \pi[(i+1)^2 \, \Delta r^2 - i^2 \, \Delta r^2] = 2\pi \, \Delta r^2 (i + 0{\cdot}5) \,. \tag{32}$$

As the mean area of the phase interface during the solidification of the shell the logarithmic mean area is the most suitable:

$$F/L = 2\pi \frac{(i+1)\,\Delta r - i\,\Delta r}{\ln \dfrac{i+1}{i}} = \frac{2\pi \, \Delta r}{\ln \dfrac{i+1}{i}} \,. \tag{33}$$

For the shells $n = 2$ to $n = N - 1$, the value B is then

$$B = (i + 0{\cdot}5) \ln \frac{i + 1}{i} . \tag{34}$$

which is valid for the N-th shell, too, in the case of the body division without the fictious half-shell. In the other case, the relation (35) can be derived in the same way as before:

$$A = (N - 0{\cdot}75) \ln \frac{N - 0{\cdot}5}{N - 1} . \tag{35}$$

For the central shell Longwell[1] recommends to take for the computation of the mean area the inner radius $r = (0{\cdot}1 \approx 0{\cdot}11)\,\Delta r$ instead of $r = 0$. The value of C is then

$$C = 1{\cdot}25 \tag{36}$$

c) *The sphere.* The volume increment of the phase g in this case is

$$\Delta V = \tfrac{4}{3}\pi\left[(i + 1)^3\,\Delta r^3 - i^3\,\Delta r^3\right] = \tfrac{4}{3}\pi\,\Delta r^3(3i^2 + 3i + 1) . \tag{37}$$

As the mean heat transfer area during the solidifying of the shell the geometric mean of the boundary areas is taken:

$$F = 4\pi(i + 1)\,\Delta r \,.\, i\,\Delta r = 4\pi\,\Delta r^2 i(i + 1) . \tag{38}$$

The value of B is then

$$B = (3i^2 + 3i + 1)/(3i^2 + 3i) \tag{39}$$

and in the case of the division of the sphere without the fictious half-shell is B valid for shells $n = 2$ to $n = N$. Otherwise it can be derived for the N-th shell the equation

$$A = \left[(N - 0{\cdot}5)^3 - (N - 1)^3\right]/3(N - 0{\cdot}5)(N - 1) . \tag{40}$$

For the calculation of the mean heat transfer area in the first shell, the inner radius is taken similarly as in the case of the cylinder $r = (0{\cdot}10 \div 0{\cdot}11)\,\Delta r$, what gives for C

$$C = 3 . \tag{41}$$

Course of Computation

During the simple cooling of the body (phase f or g), the temperature distribution in the body is determined following the Eqs (*12*, *16*, *22*, *23*), eventualy Eq. (*26*) using the appropriate physical constants for each phase. During the solidifying of the n-th shell, the temperature distribution is determined by the same equations as well as the Eq. (*6a*), which is valid just from the beginning of the solidification of each shell, and at the same time the values of $f(p)$ are computed:
for $i = N - 1$:

$$f(p) = AK/M_g - \sum_{j=1}^{p} \left[t_i + (k_f/k_g)(t_i - t_{i-1,j}) - t_{i+1,j}\right], \tag{42}$$

for $1 \leqq i \leqq N - 2$:

$$f(p) = BK/M_g - \sum_{j=1}^{p} \left[t_i + (k_f/k_g)(t_i - t_{i-1,j}) - t_{i+1,j}\right], \tag{43}$$

for $i = 0$:

$$f(p) = CK/M_g - \sum_{j=1}^{p} (t_i - t_{1,j}). \tag{44}$$

If $f(p) = 0$, then $p = P$ and the computation of the solidification of another shell follows. If $f(p-1) > 0$ and $f(p) < 0$, then the number P of the time intervals of the solidification of the shell as well as the temperature distribution in the body in the time of complete solidification of the shell are determined by linear interpolation. This final temperature distribution is the starting distribution for the computation of the solidification of the shell $n - 1$ and so on, untill the whole body is solid.

The stability of this solution for $M = 0{\cdot}5$ for all three body shapes concerned is guaranteed by the use of implicit Eq. (*16*).

During the solidification it is necessary to choose the same time interval for both phases. It follows then from the definition of the modulus M, that if $a_f \neq a_g$, it is necessary to choose for each phase either different division ($\Delta r_f \neq \Delta r_g$) or different M ($M_f \neq M_g$). In the present paper the same division Δr and different M have been used in that way, that for the phase with greater thermal diffusivity a the M can have the highest value $M = 0{\cdot}5$, so that $M < 0{\cdot}5$ for the other phase with a smaller a and the computation is stable for both phases. This way of solution is believed to be simpler and more suitable than that of Longwell[1].

Computing with dimensionless quantities, the dimensionless time increment ΔT cannot be the same for both phases, as it contains the thermal diffusivity a. In the course of the computation with the aid of derived equations, the temperature change is computed with the same time interval for both phases. If the value of a is known

that has been used for the dimensionless interpretation of the time, it is easy to determine the real time of the temperature change and solidification. In the present paper the a_f has been used for the cooling of the phase f, and a_g for the solidification and the cooling of the phase g.

To perform computations following the proposed method, a programme for the computer National Elliot 803B in autocode 3 has been elaborated. The programme enables for any Biot number the computation of simple cooling or heating as well as that of the case with the change of phase and with the interface movement, the computation with or without the fictious auxiliary half shell, with dimensional or dimensionless quantities, with any initial temperature distribution in the body and with constant or variable heat transfer coefficient h and temperature of the ambient medium. Output are the time, the temperature at the surface and in the central plane (axis, center) of the body during the cooling and the time of solidification and the temperatures at each dividing area in the moment of the total solidification of each shell during the change of phase.

EXAMPLES

1. *Simple cooling.* Grigull[6] proposed a new method to solve the temperature field in the body at very small times of cooling or heating and demonstrated its use having calculated the surface temperature of a concrete plate after its convective cooling (Bi = 7·2) at the dimensionless time $T = 0{\cdot}00697$. The result of the calculation following his method has been $T = 0{\cdot}567$. The results of numerical computation following the method of this work with various conditions of the body and time division have been tabulated in Table I.

With the division of the body in $N = 30$ shells and using the fictious half-shell, a sufficiently accurate result has been obtained, what has not been achieved even with 40 shells without the use of the half-shell.

2. *Freezing of a saturated liquid in cylinder.* The solidification time of a saturated liquid in the form of an infinitely long cylinder is given together with the results of Tao's[2] computation in

TABLE I

Surface Temperature of a Concrete Plate after a Cooling Time $T = 0{\cdot}00697$

N	$\Delta T . 10^4$	Surface temperature of the plate	
		half-shell used	without half-shell
10	13·94	0·5568	0·4787
25	6·970	0·5663	0·5325
30	5·362	0·5668	0·5380
40	3·168	0·5671	0·5441

Table II. It is evident that the solidification times of single shells as well as the time of complete solidification are slightly higher than that of Tao.

3. *Freezing of a liquid initialy at higher than fusion temperature.* The solidification time of such a liquid evidently must increase. This encrease will be the higher, the higher will be (for the same initial temperature) the thermal diffusivity of the solid in comparison with the liquid phase. The time dependence of the interface position and the temperature in the central plane (axis, center) of a plate, cylinder and sphere for the case of the same thermal diffusivity of both phases and for the case, when the thermal diffusivity of the solid phase is three times greater than that of the liquid phase, is given in Table III and in Fig. 1. The temperature of solidification is $t_s = 0{\cdot}8$, the initial temperature $t_0 = 1$. The dimensionless time has been calculated with the aid of the thermal diffusivity of the solid phase.

DISCUSSION

The computational programme prepared on the basis of the derived equations enables the computation of the temperature field and the interface position in an infinite plate of finite thickness, infinitely long cylinder and sphere with a high

TABLE II

The Freezing Time of an Infinitely Long Cylinder
$N = 40$, $K = 1$, $1/\mathrm{Bi} = 0$, $t_0 = t_s = 1$.

r	T	
	this paper	ref.[2] [a]
0·975	0·000625	0·00040
0·950	0·001852	0·00164
0·925	0·003892	0·00367
0·900	0·006698	0·00647
0·800	0·02526	0·02499
0·700	0·05457	0·05405
0·600	0·09323	0·09254
0·500	0·1396	0·13889
0·400	0·1918	0·19089
0·300	0·2473	0·24618
0·200	0·3030	0·30153
0·100	0·3539	0·35217
0·075	0·3650	0·36314
0·050	0·3750	0·37297
0·025	0·3835	0·38123
0·000	0·3904	0·38698

[a] After the complete tabulation of computation results that has been obtained by favor of Tao (*cf.*[2]).

TABLE III

The Freezing of a Plate, Cylinder, and Sphere
$N = 10$, $K = 1$, $\mathrm{Bi} = 100$, $t_0 = 1$, $t_s = 0{\cdot}8$.

r	T					
	plate		cylinder		sphere	
	$a_g = a_f$	$a_g = 3a_f$	$a_g = a_f$	$a_g = 3a_f$	$a_g = a_f$	$a_g = 3a_f$
0·9	0·0157	0·0147	0·0157	0·0147	0·0156	0·0147
0·8	0·0481	0·0444	0·0452	0·0426	0·0427	0·0410
0·7	0·1014	0·0941	0·0896	0·0853	0·0812	0·0791
0·6	0·1743	0·1609	0·1447	0·1405	0·1262	0·1256
0·5	0·2621	0·2476	0·2062	0·2051	0·1748	0·1768
0·4	0·3592	0·3522	0·2721	0·2754	0·2253	0·2293
0·3	0·4660	0·4696	0·3414	0·3471	0·2756	0·2803
0·2	0·5867	0·5946	0·4112	0·4172	0·3231	0·3278
0·1	0·7230	0·7309	0·4772	0·4831	0·3662	0·3708
0·0	0·8749	0·8828	0·5363	0·5422	0·4276	0·4322

accuracy. It is suitable for the computation of processes with the boundary condition of 1st and 3rd kind, with constant or variable heat transfer coefficient and temperature of the cooling or heating medium and with different (but constant) physical properties of both phases. The use of the auxiliary half-shell increases the accuracy of the results and is therefore advisable at lower Biot numbers ($\mathrm{Bi} < 60$). For higher Bi, however,

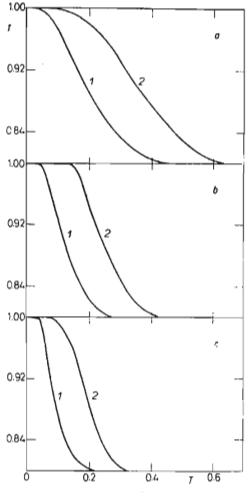

FIG. 1

The Time Dependence of the Central Temperature of the Plate (*a*), Cylinder (*b*), and Sphere (*c*) (Cooling and Solidifying of a Superheated Liquid)

$t_0 = 1$, $t_s = 0{\cdot}8$, $\mathrm{Bi} = 100$, $K = 1$, $N = 10$. Curves: 1 $a_g = a_f$; 2 $a_g = 3a^f$.

the grid used should have to be very fine (more than 30 shells), what would require a long time of computation. Computing the simple cooling or heating, the division of the body in $N = N_{min} + 5$ shells (with the use of the half shell) gives for $T > 0{\cdot}3$ practically exact results (for N_{min} see Eq. (*21*)). Division in $N > 30$ shells gives exact results even for very small times, as has been demonstrated by Example 1. The results of the computation under various conditions show the higher precision of the computation with the half-shell. Nevertheless, the division $N > 20$ is for $T \geqq 0{\cdot}3$ and $Bi > 60$ sufficiently accurate for practical purposes even without the use of the half-shell.

The solidification time of the infinitely long cylinder from the Example 2 is longer than the time calculated by Tao[2]. This difference is caused partly by different way of the computation of the solidification time of the outer shell, partly – and above all – by the use of the logarithmic mean of the heat transfer areas during the solidification of single shells and by the method used in this paper to compute the time of solidification of the central shell.

As the use of logarithmic (geometric) mean of heat transfer areas in the case of cylindrical (spherical) geometry is the most suitable way of approximation, the present results can be believed to be closer to the reality; the difference between both methods is small, however.

The main contribution of this work lies in the solution of the method of computation of the temperature field, time of solidification and interface position for the bodies initially at other than the fusion temperature and with any symmetric initial temperature distribution, which takes into account heat capacities and different thermal properties of both phases.

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LIST OF SYMBOLS

a	thermal difusivity
A	defined by Eqs (*35*) and (*40*)
b	shape factor; 0 for plate, 1 for cylinder, 2 for sphere
B	defined by Eqs (*34*) and (*39*)
$Bi = hR_0/k$	Biot number
c	specific heat capacity
C	defined by Eqs (*36*) and (*41*)
$e = \eta/R_0$	dimensionless coordinate of the phase interface
F	heat transfer area
h	convective heat transfer coefficient
i	number of dividing area on which lies the interface
j	number of the time interval during the solidification of a shell
k	thermal conductivity; number of time interval
$K = L/c_g(\Theta_0 - \Theta_p)$	dimensionless heat of fusion
L	heat of fusion

$M = a\,\Delta\tau/\Delta R^2 = \Delta T/\Delta r^2$ modulus of numerical solution of Eq. (*12*)
n number of coordinate interval
N number of dividing areas (number of shells)
p number of time intervals elapsed from the start of solidification of a shell
P number of time intervals for solidification of one shell
$r = R/R_0$ dimensionless coordinate
$\Delta r = \Delta R/R_0$ dimensionless coordinate interval (thickness of a shell)
R coordinate with the origin in the central plane of plate, axis of cylinder or center of sphere
R_0 half-plate thickness, radius of cylinder or sphere
$t = (\Theta - \Theta_p)/(\Theta_0 - \Theta_p)$ dimensionless temperature
$t_0 = 1$ dimensionless initial temperature
$T = a\tau/R_0^2$ dimensionless time (Fourier number)
V volume
Δ difference, interval
η coordinate of the phase interface
Θ temperature
Θ_0 initial temperature of the body in the central plane (axis, center)
Θ_p ambient temperature
ϱ density
τ time

Subscripts

f, g initial and newly formed phase, respectively
k number of time interval
n number of coordinate interval
s solidification (fusion) conditions

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